HSE/Math in Moscow 2015-2016// Topology 2 // Problem sheet 6

Cellular chain complexes

Question 1. Let X be an orientable compact connected surface of genus g, i.e. $X = T^2\#\cdots\#T^2$ (g times). Let us equip X with a CW-structure using the $4g$-gon model.

(a) Briefly explain why the cellular chain complex of X is as described in Lecture 8. You may also assume the integral homology of X to be known and deduce the result algebraically.

(b) Calculate $H_*(X,G)$ and $H^*(X,G)$ for $G = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Q}$. You may use the fact that $H_*(\mathcal{C}_*(X)\otimes G) \cong H_*(X,G)$ and $H^*(X,G) \cong H^*(\mathrm{Hom}(\mathcal{C}_*(X), G)$, functorially with respect to cellular maps.

Question 2. Same as question 1 for $X =$ a non-orientable compact connected surface of genus g, i.e. $X = \mathbb{R}P^2\#\cdots\#\mathbb{R}P^2$ (g times). (Only this time we use the $2g$-gon model to construct a CW-structure.)

Question 3. (a) Equip $\mathbb{R}P^n$ with a CW-structure with one cell of every dimension from 0 to n so that $\mathbb{R}P^{n-1}$ becomes a subcomplex of $\mathbb{R}P^n$.

(b) Prove that the cellular chain complex $\mathcal{C}_* = \mathcal{C}_*(\mathbb{R}P^n)$ of $\mathbb{R}P^n$ has $\mathcal{C}_i = \mathbb{Z}$ for all $i = 0,\dots,n$ and $\partial_i : \mathcal{C}_i \to \mathcal{C}_{i-1}$ is zero for $n = 0$ and for odd n and is multiplication by 2 for all even $n \neq 0$. You may assume the integral homology of $\mathbb{R}P^n$ to be known.

(c) Calculate $H_*(\mathbb{R}P^n, G)$.

Question 4. Calculate the integral homology of $\mathbb{R}P^2\times\mathbb{R}P^2$

(a) Using the Künneth formula for homology $H_n(X\times Y, G) \cong \bigoplus_{p+q=n} H_p(X, H_q(Y,G))$.

(b) Using cellular chain complexes and the fact that $\mathcal{C}_*(X\times Y) \cong \mathcal{C}_*(X)\otimes\mathcal{C}_*(Y)$ where X, Y are CW-complexes.

(c) Is it true that for all n we have $H_n(\mathbb{R}P^2\times\mathbb{R}P^2) \cong \bigoplus_{p+q=n} H_p(\mathbb{R}P^2)\otimes H_q(\mathbb{R}P^2)$?

Borsuk-Ulam

Question 5. Let us pretend for a moment that we did discuss the Borsuk-Ulam theorem on Friday. Recall that the theorem says that the degree of an odd self-map of S^n is odd. Let $f : S^n \to \mathbb{R}^n$ be a continuous map and set $g(x) = f(x) - f(-x)$ for $x \in S^n$.

(a) Suppose g has no zeroes. Use g to construct an odd map $h : S^n \to S^{n-1}$.

(b) Using the Borsuk-Ulam theorem show that there are no odd maps $S^n \to S^{n-1}$. [Hint: restrict to the equator.]

(c) Deduce that g has a zero, and so that there is an $x \in S^n$ such that $f(x) = f(-x)$

Problems for discussion

None this week: we really have to talk a bit about the Borsuk-Ulam theorem before we move on to other things.